Two body problem and its reduction to one body problem and its Solution:—

Let us Consider a conservative system of two mass points m_1 and m_2. Such a system has six degrees of freedom because the motion of the system consisting of m_1 and m_2 can be thought as made up of motion of centre of mass and motion about the centre of mass. Let us choose them to be the three components of $\vec{R}$, the position vector of the centre of mass, and three components of difference vector

$$\vec{r} = (\vec{r}_1 - \vec{r}_2) \quad \text{——— (1)}$$

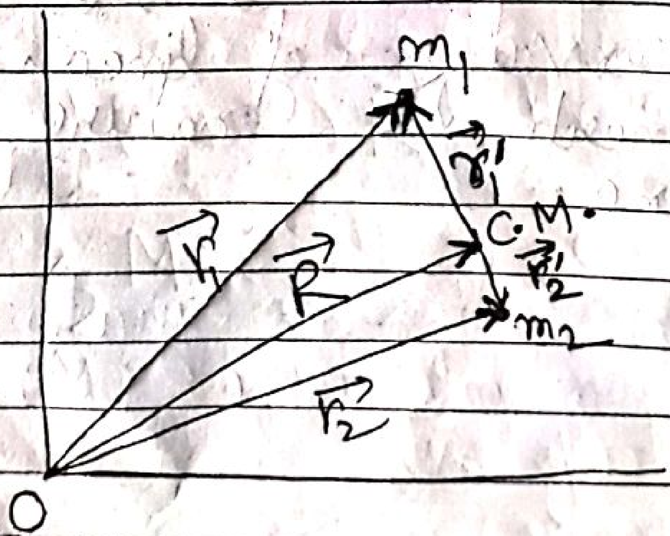

Fig. Relation among the various co-ordinates

The Lagrangian for such a system can be written as

$$L = T(\dot{R}, \dot{r}) - V(r)$$

where, $$T(\dot{R}, \dot{r}) = \frac{1}{2}(m_1 + m_2)\dot{R}^2 + T'$$

which expresses the kinetic energy as the sum of kinetic energy of motion of the centre of mass, plus the kinetic energy of motion about the centre of mass T'. It has been assumed that V is purely a function of $\vec{r}$, the separation between two particles. T' is given by

$$T' = \frac{1}{2} m_1 \dot{\vec{r}}_1'^2 + \frac{1}{2} m_2 \dot{\vec{r}}_2'^2$$

where $\vec{r}_1'$ and $\vec{r}_2'$ are the position vectors with respect to the centre of mass and the difference vector is

$$\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{r}_1' - \vec{r}_2' \quad \text{——} \quad (2)$$

We know that a body balances about the centre of mass and hence $\sum_i m_i \vec{r}_i' = 0$. Therefore,

$$m_1 \vec{r}_1' + m_2 \vec{r}_2' = 0$$

or, $$\vec{r}_1' = -\left(\frac{m_2}{m_1}\right)\vec{r}_2' \quad \text{——} \quad (3)$$

Putting eq. (3) in eq. (2), we get

$$\vec{r} = \vec{r}_1' + \left(\frac{m_1}{m_2}\right)\vec{r}_1'$$

$$\vec{r} = \left(\frac{m_1+m_2}{m_2}\right)\vec{r}_1'$$

or, $$\vec{r}_1' = \left(\frac{m_2}{m_1+m_2}\right)\vec{r}$$

also, $$\vec{r}_2' = -\left(\frac{m_1}{m_1+m_2}\right)\vec{r}$$

Kinetic energy is,

$$T = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}m_1\dot{\vec{r}}_1'^2 + \frac{1}{2}m_2\dot{\vec{r}}_2'^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2\dot{\vec{r}}^2 + \frac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\right)^2\dot{\vec{r}}^2,$$

after putting values of $\dot{\vec{r}}_1'$ and $\dot{\vec{r}}_2'$ from eq. (4). Thus

$$T = \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 \cdot m_2}{m_1+m_2}\dot{\vec{r}}^2$$

$$= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$$

where $M = m_1 + m_2$ is the total mass of the system and

$$\mu = \frac{m_1 \cdot m_2}{m_1+m_2}$$

is the reduced mass.

The Lagrangian can be expressed as,

Scanned with CamScanner

$$L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{r}^2 - V(r)$$

from which we note that the three components of $\vec{R}$ do not appear in the Lagrangian L and consequently are cyclic.

Lagrange's equations of motion in terms of the two variables R and r will be

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{R}}}\right) - \frac{\partial L}{\partial R} = 0 \quad \text{—— (6)}$$

and,
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \quad \text{—— (7)}$$

Putting values of the derivative from eq. (5) into eqs. (6) and (7), we get

$$\frac{d}{dt}(M\dot{\vec{R}}) = 0 \quad \text{—— (8)}$$

and,
$$\frac{d}{dt}(\mu \dot{r}) + \frac{\partial}{\partial r} V(r) = 0 \quad \text{—— (9)}$$

Eq. (8) gives that momentum conjugate to R is constant. That is,

$$M\dot{\vec{R}} = \text{Constant}$$

or,
$$\dot{\vec{R}} = \text{Constant}$$

That is, velocity with which the

Date: ________ Page: ________

centre of mass moves is constant. Eq. 9 gives

$$\mu \ddot{r} = -\frac{\partial}{\partial r} V(r)$$

$$= f(r),$$

representing equation of motion for the system under consideration, with interparticle distance as r.

Thus none of the equations of motion for $\vec{r}$ will contain terms involving $\vec{R}$ to $\dot{\vec{R}}$. Consequently, we can ignore the first term in eq. 5 and write

$$L = \frac{1}{2} \mu \dot{\vec{r}}^2 - V(r), \qquad \text{——— (10)}$$

which is effective in describing the motion of the components of r.

But Lagrangian given by eq. 10 is the same as would be in the case of single particle of mass μ, moving at a distance r from a fixed centre of force which gives rise to the potential energy $V(r)$ and hence this type of two body problem can always be reduced to the equivalent one body problem.